U-Substitution

The general form of an integrand which requires U-Substitution is $\int f(g(x))g'(x)dx$. This can be rewritten as $\int f(u)du$.

A big hint to use U-Substitution is that there is a composition of functions and there is some relation between two functions involved by way of derivatives.

Example 1

$\int \sqrt{3x+2}dx$

Let $u = 3x+2$. Then $du = 3dx$ and thus $dx = \frac{1}{3}du$. We then consider $\int \sqrt{u}(\frac{1}{3})du$.

$\frac{1}{3}\int \sqrt{u}du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{2}{9}u^{3/2} + C$

Next we must make sure to have everything in terms of x like we had in the beginning of the problem. From our previous choice of u, we know $u = 3x+2$.

So our final answer is $\frac{2}{9}(3x+2)^{3/2} + C$.

For indefinite integrals, always make sure to switch back to the variable you started with.

Example 2

$\int_1^2 x^3\cos(x^4+3)dx$

Let $u = x^4+3$. So $du = 4x^3dx$. Then $\frac{1}{4}du = x^3dx$

From here we have two options. We can either switch back to x later and plug in our bounds after or we can change our integral bounds along with our U-Substitution and solve.

Option 1:

If we do not change our bounds, we have $\int_a^b \frac{1}{4}\cos(u)du$. Note that we use a and b as placeholders for now.

$\int_a^b \frac{1}{4}\cos(u)du = \frac{1}{4}\sin(u)|_a^b$

By substituting back for x using $u = x^4+3$, we have $\frac{1}{4}\sin(x^4+3)|_1^2$.

Note, we can put our original bounds back once we have everything in terms of x. Thus we have $\frac{1}{4}\sin(2^4+3) - \frac{1}{4}\sin(1^4+3) = \frac{1}{4}\sin(19) - \frac{1}{4}\sin(4)$

Option 2:

If we change our bounds, we need $u = x^4+3$. If $x = 1$ then $u = 4$. If $x = 2$ then $u = 19$. Now our problem becomes:

$\int_4^{19} \frac{1}{4}\cos(u)du = \frac{1}{4}\sin(u)|_4^{19} = \frac{1}{4}\sin(19) - \frac{1}{4}\sin(4)$

In both options we reach the same answer.

Example 3

$\int \sqrt{1+x^2}x^5dx$

Let $u = 1+x^2$. Then $du = 2xdx$ and $\frac{1}{2}du = xdx$. Because we have more x's than our substitution takes care of, we have an additional step. $u = 1+x^2$ tells us that $x^2 = u-1$.

$\int \sqrt{1+x^2}x^5dx = \int \sqrt{1+x^2}x^2x^2xdx = \int \sqrt{u}(u-1)(u-1)\frac{1}{2}du = \frac{1}{2}\int \sqrt{u}(u-1)^2du$

$= \frac{1}{2}\int \sqrt{u}(u^2-2u+1)du = \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})du = \frac{1}{2}[\frac{u^{7/2}}{7/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}] + C$

$= \frac{u^{7/2}}{7} - 2\frac{u^{5/2}}{5} + \frac{u^{3/2}}{3} + C = \frac{(1+x^2)^{7/2}}{7} - 2\frac{(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C$

Integration by Parts
The general form of an integrand which requires integration by parts is $\int f(x)g'(x)dx$. Thus it has the form $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)$dx.

Alternatively, we can use $\int udv = uv - \int vdu$

Typically, when deciding which function is u and which is dv we want our u to be something whose derivative becomes easier to deal with.

Example 4
$\int x \sin x dx$
We choose our $u = x$ since it's derivative becomes easier than $\sin x$.
Then $u = x,\ du = dx,\ dv = \sin x dx,$ and $v = -\cos x$. Following the formula, we have
$\int udv = uv - \int vdu = x(-\cos x) - \int(-\cos x)dx = -x\cos x + \sin x + C$

Example 5
$\int \ln x dx$
We choose $u = \ln x$ since $\ln x$ becomes easier to work with when we take its derivative. Note that the integrand has another function present, a constant of 1. We can rewrite the problem as $\int \ln x \cdot 1dx$.
So $u = \ln x,\ du = \frac{1}{x},\ dv = 1dx,$ and $v = x$.
$\int udv = uv - \int vdu = \ln(x) \cdot x - \int x \cdot \frac{1}{x}dx = x\ln x - \int 1dx = x\ln x - x + C$

Example 6
$\int t^2 e^t dt$

Some problems, such as this one, require two steps of integration by parts. Looking at our functions, e^t, does not have an easier derivative or antiderivative to work with. So we choose $u = t^2$. Then $du = 2tdt,\ dv = e^t,$ and $v = e^t$. Thus we have
$\int udv = uv - \int vdu = t^2e^t - \int e^t(2t)dt$

The integral that results requires integration by parts once more. We focus on $\int e^t(2t)dt$.
For the next step we will use different variables just so we do not confuse them with the previous step.
Let $w = 2t$ for $\int wdz = wz - \int zdw$. Then $dw = 2dt,\ dz = e^t,$ and $z = e^t$.
$\int wdz = wz - \int zdw = 2te^t - \int e^t(2)dt = 2te^t - 2e^t + C$

Combining the two parts we have:
$\int udv = uv - \int vdu = t^2e^t - \int e^t(2t)dt = t^2e^t - [2te^t - 2e^t + C] = t^2e^t - 2te^t + 2e^t + C$