Calculus I - Intuition with Limits

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Suggested Approaches to Limits (not including ∞)

Direct Substitution (Plugging in the Value) (1)
(i) If you get a number in the end then you are done.
(ii) If there is vertical asymptote (a dividing by 0 issue), then try another approach.

Factoring (2)
Factoring may fix the issue of dividing by 0 if things cancel. Afterwards, plug in the value.

Multiplying by the Conjugate (3)
Similarly, multiplying by the conjugate may fix the issue of dividing by 0. Plug in the value afterwards.

Evaluating Limits Involving ∞

For Rational Functions, as $x \to \infty$ and $x \to -\infty$, Check the Degree (4)
(i) If the numerator and denominator are the same degree then the limit is the ratio of the leading coefficients.
(ii) If the denominator is a higher degree than the numerator then the limit is 0.
(iii) If the numerator is a higher degree than the denominator, then the limit is either ∞ or $-\infty$.

Limits Involving a Vertical Asymptote (5)
If there is still a vertical asymptote after checking the methods above then the left-hand limit and right-hand limit must be calculated.
(i) If the left-hand and right-hand limits are equal then the two-sided limit matches.
(ii) If the left-hand and right-hand limits are NOT the same then the limit does not exist.

Special Cases

Trigonometric Functions (6)
It may be helpful to remember commonly occurring limits involving trigonometric functions. Two common limits are:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{1 \quad \cos(x)}{x} = 0$$

The Squeeze Theorem (7)
If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a), the limits of f and h both exist as x approaches a, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Looking at Behavior and Composition of Functions (8)
The previous methods will help solve **MOST** problems. Some problems may involve looking at the behavior of the functions involved or evaluating limits within a composition of functions.

Examples

Method	Problem	Solution
(1)	$\lim\limits_{x\to 2} x^3$	$\lim\limits_{x\to 2} x^3 = (2)^3 = 8$
(2)	$\lim\limits_{x\to -1} \frac{x^2-x-2}{x^2-2x-3}$	$\lim\limits_{x\to -1} \frac{(x-2)(x+1)}{(x-3)(x+1)} = \lim\limits_{x\to -1} \frac{(x-2)}{(x-3)} = \frac{((-1)-2)}{((-1)-3)} = \frac{3}{4}$
(3)	$\lim\limits_{x\to 9} \frac{\sqrt{x}-3}{x-9}$	$\lim\limits_{x\to 9} \frac{\sqrt{x}-3}{x-9}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right) = \lim\limits_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim\limits_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$
(4)	$\lim\limits_{x\to\infty} \frac{3x^2+x-2}{5x^2-2x+3}$	$\lim\limits_{x\to\infty} \frac{3x^2+x-2}{5x^2-2x+3}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim\limits_{x\to\infty} \frac{3+\frac{1}{x}-\frac{2}{x^2}}{5-\frac{2}{x}+\frac{3}{x^2}} = \frac{3}{5}$
(5)	$\lim\limits_{x\to 2} \frac{1}{x-2}$	$\lim\limits_{x\to 2^-} \frac{1}{x-2} = -\infty \neq \lim\limits_{x\to 2^+} \frac{1}{x-2} = \infty \quad \therefore \lim\limits_{x\to 2} \frac{1}{x-2}$ Does Not Exist
(6)	$\lim\limits_{x\to 0} \frac{\sin(2x)}{5x}$	$\lim\limits_{x\to 0} \frac{\sin(2x)}{5x} = \lim\limits_{x\to 0} \frac{\sin(2x)}{5x} \cdot \frac{2}{2} = \lim\limits_{x\to 0} \frac{2}{5} \cdot \frac{\sin(2x)}{2x} = \frac{2}{5} \cdot \lim\limits_{x\to 0} \frac{\sin(2x)}{2x} = \frac{2}{5}$
(7)	$\lim\limits_{x\to\infty} \frac{\sin(x)}{x}$	$-1 \leq \sin(x) \leq 1 \implies -\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$ $\implies \lim\limits_{x\to\infty} -\frac{1}{x} \leq \lim\limits_{x\to\infty} \frac{\sin(x)}{x} \leq \lim\limits_{x\to\infty} \frac{1}{x}$ $\implies 0 \leq \lim\limits_{x\to\infty} \frac{\sin(x)}{x} \leq 0$ $\implies \lim\limits_{x\to\infty} \frac{\sin(x)}{x} = 0$
(8)	$\lim\limits_{x\to 0^+} \tan^{-1}(\frac{1}{x})$	$\lim\limits_{x\to 0^+} \tan^{-1}(\frac{1}{x}) = \tan^{-1}\left(\lim\limits_{x\to 0^+} (\frac{1}{x})\right) = \lim\limits_{y\to\infty} \tan^{-1} y = \frac{\pi}{2}$
(8)	$\lim\limits_{x\to 1} e^{x^2-x}$	$\lim\limits_{x\to 1} e^{x^2-x} = e^{\left(\lim\limits_{x\to 1} x^2-x\right)} = e^{(1^2-1)} = e^0 = 1$

Note: In the last 2 examples we are able to pull limits in and out of compositions of functions because e and $\tan^{-1}$ are continuous functions.