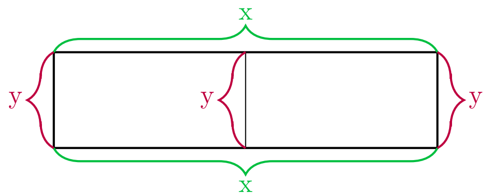
Calculus

Optimization

Problem 1: Building a Fence

A farmer wants to fence in an area of 13.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

1 Draw a picture and label your variables:



2 Create two equations (cost and area) based on the given data and your picture:

$$C = 2x + 3y \tag{1}$$

$$xy = 13,500,000 \quad ft^2 \tag{2}$$

3 Solve equation (2) for one of the variables:

$$xy = 13,500,000$$
$$x = \frac{13,500,000}{y} \tag{3}$$

4 Substitute x into equation (1):

$$C(y) = 2\left[\frac{13,500,000}{y}\right] + 3y$$
$$= \frac{27,000,000}{y} + 3y$$

5 Take the derivative of the equation found in step 4, set this equal to zero and solve:

$$C'(y) = \frac{27,000,000}{y^2} + 3$$
$$0 = \frac{27,000,000}{y^2} + 3y^2$$
$$0 = 3\left[\frac{9,000,000}{y^2} + 1\right]$$
$$0 = \frac{9,000,000}{y^2} + 1$$
$$y^2 = 9,000,000$$
$$y = 3,000$$

6 Substitute y into to equation (3):

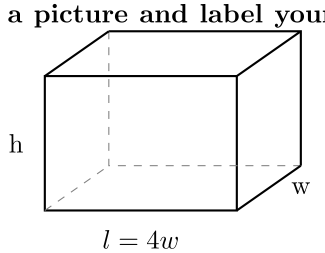
$$x = \frac{13,500,000}{3000}$$
$$x = 4500$$

This means that the farmer should build his fence with a length of $3000ft^2$ and a width of $4500ft^2$ to use the shortest total length of fence for an area of $13,500,000ft^2$.

Problem 2: Volume

We want to construct a box whose base length is 4 times the base width. The material used to build the top and bottom cost \$6.00/ft^2 and the material used to build the sides cost \$21.60/ft^2. If the box must have a volume of 1296 ft^3, determine the dimensions that will minimize the cost to build the box.

1 Draw a picture and label your variables:



2 Create two equations (cost and volume) based on the given data and your picture:

$$V = lwh = 4w^2h = 1296 \quad ft^3 \tag{1}$$

$$C = 6.00(2lw) + 21.60(2wh + 2lh)$$
$$C = 6.00(8w^2) + 21.60(2wh + 8wh)$$
$$C = 48w^2 + 216wh \tag{2}$$

3 Solve equation (1), $1296 = 4w^2h$, for h:

$$1296 = 4w^2h$$
$$h = \frac{1296}{4w^2}$$

4 Substitute h into equation (2):

$$C(w) = 48w^2 + 216w\left(\frac{1296}{4w^2}\right)$$
$$C(w) = 48w^2 + \frac{69984}{w}$$

5 Take the derivative of the equation found in step 4, set this equal to zero and solve:

$$C(w) = 48w^2 + 69984w^{-1}$$
$$C'(w) = 96w - 69984w^{-2}$$
$$0 = \frac{96w^3 - 69984}{w^2}$$

Setting the denominator to zero gives a width of zero. We can ignore this because a box cannot have zero width.

$$96w^3 = 69984$$
$$w^3 = 729$$
$$w = 9$$

6 Substitute w into equation (1):

$$1296 = 4(9)^2h$$
$$1296 = 324h$$
$$4 = h$$

Using both w and h in $1296 = lwh$ again gives:

$$1296 = l \times 4 \times 9$$
$$1296 = 36l$$
$$36 = l$$

This means that the the box should be constructed to be $36ft$ long, $9ft$ wide and $4ft$ high to minimize the cost.

As a bonus, we can use these values to find how much this box would cost to make!

$$C(9) = 48(9)^2 + 216(9)(4)$$
$$= 3888 + 7776$$
$$= 11664$$

This means it would cost at least \$11,664 to construct this box.