Math 264 California State University SAN MARCOS

1 Systems of Linear Equations to Matrices

Suppose we have the following system of equations:
$x - y + 2z = 5$
$2x - 2y + 4z = 10$
$3x - 3y + 6z = 15$

We can translate the system into an augmented matrix. The following abbreviation can denote an augmented matrix.

$$\begin{bmatrix} a_1x & b_1y & c_1z & d_1 \\ a_2x & b_2y & c_2z & d_2 \\ a_3x & b_3y & c_3z & d_3 \\ a_nx & b_ny & c_nz & d_m \end{bmatrix}$$

1. Identify coefficients (a, b, c).
$a_1 = 1, b_1 = -1, c_1 = 2$
$a_2 = 2, b_2 = -2, c_1 = 4$
$a_3 = 3, b_1 = -3, c_1 = 6$

2. Identify variables (x, y, z).
$x =?$
$y =?$
$z =?$

3. Identify constants (d).
$d_1 = 5$
$d_2 = 10$
$d_3 = 15$

4. Construct an augmented matrix from what we have identified.

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{bmatrix}$$

2 Row Echelon & Reduced Row Echelon Form

An m x n matrix is in row echelon form if
1. Any row consisting of zeros is at the bottom.
2. In each non-zero row, the first non-zero entry (leading entry) is in the column to the left of any leading entries below it.
For example, the following matrix is in row echelon form:

$$\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{bmatrix}$$

An m x n matrix is in reduced row echelon form if
1. It is in row echelon form
2. The leading entry in each non-zero row is 1 (leading 1).
3. Each column containing a leading 1 has zeros everywhere else.
For example, the following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{bmatrix}$$

3 Elementary Row Operations

1. Interchange two rows.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & k & l & m \end{bmatrix} \rightarrow \begin{bmatrix} e & f & g & h \\ a & b & c & d \\ i & k & l & m \end{bmatrix} \qquad R_1 \iff R_2$$

2. Multiply a row by a non-zero constant.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & k & l & m \end{bmatrix} \rightarrow \begin{bmatrix} 2a & 2b & 2c & 2d \\ e & f & g & h \\ i & k & l & m \end{bmatrix} \qquad 2R_1$$

3. Add a multiple of a row by to another row.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & k & l & m \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i-2a & k-2b & l-2c & m-2d \end{bmatrix} \qquad R_3 - 2R_1$$

4 Gaussian & Gauss-Jordan Elimination

Gaussian & Gauss-Jordan Elimination are row reduction algorithms that solve for a system of linear equations. In simplest terms:
Gaussian Elimination:
1. Convert the matrix to row echelon form (Elementary Row Operations).
2. Perform back substitution.
3. Find the solution set.
Gauss-Jordan Elimination:
1. Convert the matrix to reduced row echelon form. (Elementary Row Operations).
2. Find the solution set.
Types of Solutions:
1. Unique Solution (Consistent).

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

2. No Solution (Inconsistent).

$$\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow 0 \neq 3$$

3. Infinite Solutions (Consistent).

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5 Formulas

Norm of a vector: $||v|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$

Unit Vector: $\frac{1}{||v||} v$

Dot Product: $u \cdot v = u_1 v_1 + u_2 v_2 + ... + u_n v_n$

Distance between two vectors: $d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_n - v_n)^2}$

Angle between two vectors: $\cos(\theta) = \frac{u \cdot v}{||u|| \, ||v||}$

Projections: $proj_u(v) = (\frac{u \cdot v}{||v||^2}) v$

Distance between a Point and a Plane: $D = \frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}$

Inverse of 2 x 2 Matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of m x n Matrix: $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \begin{bmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{bmatrix} rref \rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & j & k & l \\ 0 & 1 & 0 & m & n & o \\ 0 & 0 & 1 & p & q & r \end{bmatrix} \rightarrow \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix}$$

Determinant of 2 x 2 Matrix: $det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Determinant of m x n Matrix: $det \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot det \begin{vmatrix} d & f \\ g & i \end{vmatrix} +$

$c \cdot det \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Rotation Matrix: $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Transpose Matrix: $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T \rightarrow \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$