Section 4.5: Absolute Maxima and Minima (Absolute Extrema)

Definition: Let c be a value in the domain of a function f. If $f(c) \geq f(x)$ for all x in the domain of f then $f(c)$ is called the ***absolute maximum*** of f. If $f(c) \leq f(x)$ for all x in the domain of f then $f(c)$ is called the ***absolute minimum*** of f.

Note: Keep in mind that c is an x-value, and $f(c)$ is y-value at $x = c$.

Procedure: Finding absolute extrema on a closed interval [a, b]

1. Check if f continuous over the interval [a, b].
2. Find the critical numbers on the interval (a, b).
3. Find $f(a), f(b)$, and evaluate f at all critical numbers found in step 2.
4. The *absolute maximum* is the largest value found in step 3.
5. The *absolute minimum* is the smallest value found in step 3.

Example: Find the absolute maximum and absolute minimum of

$$f(x) = x^3 + 3x^2 - 9x - 7 \text{ on the interval [-4, 2]}$$

1. $f(x)$ is continuous for all values of x since it is a polynomial function. So, it is continuous on [-4, 2].
2. $f'(x) = 3x^2 + 6x - 9 = 3(x-1)(x+3) = 0$

   $x - 1 = 0 \text{ or } x + 3 = 0$, so $\boldsymbol{x = 1}$ *or* $\boldsymbol{x = -3}$ are critical numbers.

3.

x	-4	-3	1	2
$f(x)$	13	20	-12	-5

4. The absolute maximum value is 20 at x = -3
5. The absolute minimum value is -12 at x = 1

Second Derivative Test for Absolute Extrema on an Interval (If there is exactly ONE critical number).

Let f be continuous on interval (a, b) with only one critical number c in the interval.

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on the open interval (a, b).

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on the open interval (a, b).

Note: Observe that in the first case, $f''(c)$ is positive, thus the function is concave up ᴗ which gives us a minimum value, and in the second case $f''(c)$ is negative, thus the function is concave down ⌢ which gives us a maximum value.

Example: Find the absolute extrema of the function on $(0, \infty)$.

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2}{x^2} - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x-2)(x+2)}{x^2} = 0$$

$x - 2 = 0 \; or \; x + 2 = 0$. So, critical numbers are $x = 2 \; or \; x = -2$. However, $x = -2$ is not in the interval $(0, \infty)$, the only critical number is $x = 2$. Note that $x = 0$ is a partition number not critical number since it 0 does not belong in the domain.

$$f''(x) = \frac{8}{x^3}$$

$f''(2) = \frac{8}{2^3} = 1 > 0$. So, since the second derivative is positive, the function is concave up, and the absolute minimum is $f(2) = 4$.

$$f(2) = 2 + \frac{4}{2} = 4$$